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Stability of H-A and T-A coupled formulations

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CERN

HTS 2020 Modelling

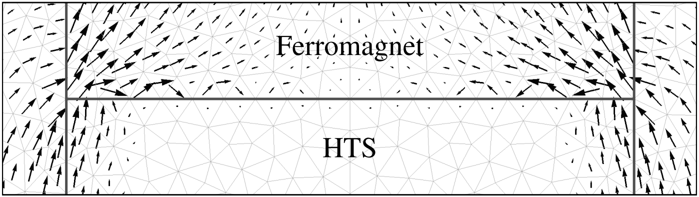
Introduction

Coupled formulations such as H-A or T-A often offer many advantages for HTS modeling:

- improved efficiency for nonlinear system resolution,
- reduced number of DOFs,
- increased flexibility,
- easier geometry definition...

However, they enter the framework of mixed formulations, thus requiring to be extremely careful regarding function spaces.

Otherwise, non-physical results must be expected:



Strong form

- Magnetodynamic (quasistatic) equations

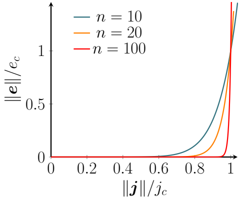
$$\operatorname{div} \boldsymbol{b} = 0, \qquad \mathbf{curl}\, \boldsymbol{h} = \boldsymbol{j}, \qquad \mathbf{curl}\, \boldsymbol{e} = -\partial_t \boldsymbol{b}.$$

- Constitutive relationships

High-temperature superconductors (HTS):

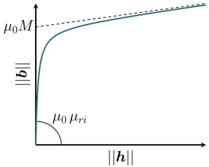
$$\boldsymbol{e} = \rho(\|\boldsymbol{j}\|)\boldsymbol{j} \quad \text{and} \quad \boldsymbol{b} = \mu_0\, \boldsymbol{h},$$

with the power law $\rho(\|\boldsymbol{j}\|) = \frac{e_c}{j_c}\left(\frac{\|\boldsymbol{j}\|}{j_c}\right)^{n-1}$.



Ferromagnetic material (FM):

$$\boldsymbol{b} = \mu(\boldsymbol{b})\, \boldsymbol{h} \quad \text{and} \quad \boldsymbol{j} = \boldsymbol{0}.$$



Dual formulations

Two classes of formulations with the finite element method:

- h-conform, e.g. h-formulation,
  - enforces the continuity of the tangential component of $\boldsymbol{h}$,
  - involves $\boldsymbol{e} = \rho\boldsymbol{j}$ and $\boldsymbol{b} = \mu\boldsymbol{h}$,
  - with $\mathbf{curl}\ \boldsymbol{h} = \mathbf{0}$ in non-conducting domain ("$\boldsymbol{h}$-ϕ"+cuts),

$$\left(\partial_t(\mu\boldsymbol{h})\ ,\boldsymbol{h}'\right)_{\Omega} + \left(\rho\, \mathbf{curl}\ \boldsymbol{h}\ ,\mathbf{curl}\ \boldsymbol{h}'\right)_{\Omega_c} - \left\langle \boldsymbol{e} \times \boldsymbol{n}\ ,\boldsymbol{h}'\right\rangle_{\Gamma_e} = 0.$$

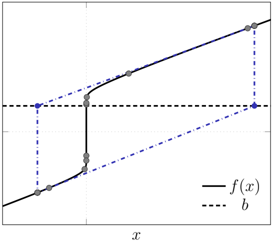
- b-conform, e.g. a-formulation,
  - enforces the continuity of the normal component of $\boldsymbol{b}$,
  - involves $\boldsymbol{j} = \sigma\boldsymbol{e}$ and $\boldsymbol{h} = \nu\boldsymbol{b}$, ($\sigma = \rho^{-1}$, $\nu = \mu^{-1}$)

$$\left(\nu\, \mathbf{curl}\ \boldsymbol{a}\ ,\mathbf{curl}\ \boldsymbol{a}'\right)_{\Omega} + \left(\sigma\, \partial_t\boldsymbol{a}\ ,\boldsymbol{a}'\right)_{\Omega_c} - \left\langle \boldsymbol{h} \times \boldsymbol{n}_{\Omega}\ ,\boldsymbol{a}'\right\rangle_{\Gamma_h} = 0.$$

Nonlinear constitutive laws involved in opposite ways $\Rightarrow$ very different numerical behaviors are expected... and observed.

Best choice for HTS only

Cycles in iterations:



In the a-formulation, the diverging slope associated with $\boldsymbol{j} = \sigma \boldsymbol{e}$ for $\boldsymbol{e} \to 0$ is really difficult to handle.

$\Rightarrow$ Among the two simple formulations, the h-formulation is much more efficient for systems with HTS:

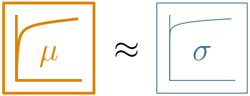
- with an adaptive time-stepping algorithm,
- solved with a Newton-Raphson method.

Dular, J., et al. (2020) TAS 30 8200113.

Ferromagnetic materials

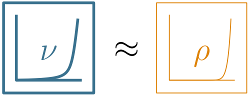
The nonlinearity is in the magnetic constitutive law.

- h-formulation the involved law is $\boldsymbol{b} = \mu \boldsymbol{h}$.



$\Rightarrow$ Often enters cycles with Newton-Raphson.
OK with fixed point, or N-R with relaxation factors but slow.

- a-formulation the involved law is $\boldsymbol{h} = \nu \boldsymbol{b}$.



$\Rightarrow$ Efficiently solved with Newton-Raphson.

The a-formulation is more appropriate for dealing with the nonlinearity, whereas for HTS, the h-formulation is best.

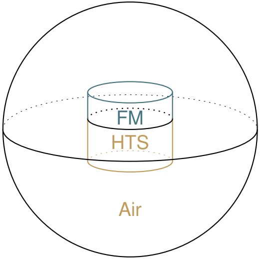
Coupled materials - h-a-formulation

Use the best formulation in each material

Decompose the domain Ω, for example into:

- $\Omega^h = \{\text{HTS, Air}\}$
- $\Omega^a = \{\text{Ferromagnet}\}$

and couple via $\Gamma_\mathrm{m} = \partial(\mathrm{FM})$:



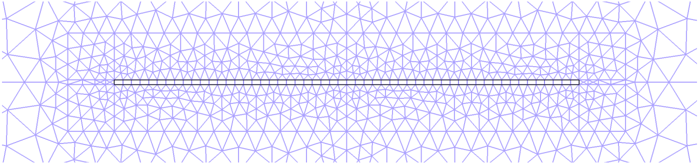
$$\begin{aligned}\left(\partial_t(\mu \boldsymbol{h})\ , \boldsymbol{h}'\right)_{\Omega^h} + \left(\rho\, \mathbf{curl}\, \boldsymbol{h}\ , \mathbf{curl}\, \boldsymbol{h}'\right)_{\Omega^h_\mathrm{c}} + \left\langle \partial_t \boldsymbol{a} \times \boldsymbol{n}_{\Omega^h}\ , \boldsymbol{h}'\right\rangle_{\Gamma_\mathrm{m}} &= 0,\\ \left\langle \boldsymbol{h} \times \boldsymbol{n}_{\Omega^a}\ , \boldsymbol{a}'\right\rangle_{\Gamma_\mathrm{m}} - \left(\nu\, \mathbf{curl}\, \boldsymbol{a}\ , \mathbf{curl}\, \boldsymbol{a}'\right)_{\Omega^a} &= 0.\end{aligned}$$

Dular, J., et al. (2020) TAS 30 8200113.
See also: Brambila R. et al, (2018) TAS 28, 5207511.

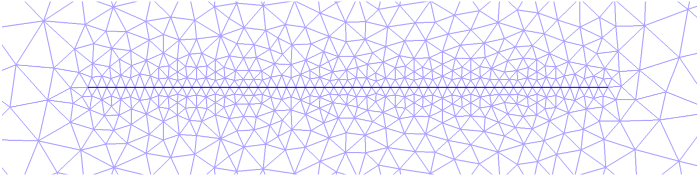
HTS tapes - t-a-formulation

To model thin superconducting tapes, two main possibilities:

1. Use the true geometry and the h-formulation with one-element across the thickness (quad., prism, hexa.).



2. Perform a thin shell approximation and model the tape as a line $\Rightarrow$ t-a-formulation.



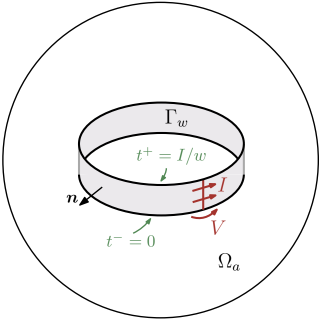
H. Zhang, et al. (2016) SUST 30, no. 2, p. 024005.

HTS tapes - t-a-formulation

Consider an HTS tape Γ_w of thickness w.

The current density is described by a current potential $\boldsymbol{t}$:

- such that $\boldsymbol{j} = \mathbf{curl}\, \boldsymbol{t}$,
- gauged by being defined along the normal of the tape, $\boldsymbol{t} = t\, \boldsymbol{n}$,
- with BC related to the total current I ($t^+ - t^- = I/w$).



$$\left(\nu\, \mathbf{curl}\, \boldsymbol{a}\ , \mathbf{curl}\, \boldsymbol{a}'\right)_{\Omega_a} - \left\langle w\, \mathbf{curl}\, \boldsymbol{t}\ , \boldsymbol{a}'\right\rangle_{\Gamma_w} = 0,$$
$$\left\langle w\, \partial_t \boldsymbol{a}\ , \mathbf{curl}\, \boldsymbol{t}'\right\rangle_{\Gamma_w} + \left\langle w\, \rho\, \mathbf{curl}\, \boldsymbol{t}\ , \mathbf{curl}\, \boldsymbol{t}'\right\rangle_{\Gamma_w} = 0.$$

Perturbed saddle point problems

h-*a*-formulation

$$\left(\partial_t(\mu \boldsymbol{h}) \,, \boldsymbol{h}'\right)_{\Omega^h} + \left(\rho\, \mathbf{curl}\, \boldsymbol{h} \,, \mathbf{curl}\, \boldsymbol{h}'\right)_{\Omega^h_{\mathrm{c}}} + \left\langle \partial_t \boldsymbol{a} \times \boldsymbol{n}_{\Omega^h} \,, \boldsymbol{h}'\right\rangle_{\Gamma_{\mathrm{m}}} = 0, \quad \forall \boldsymbol{h}' \in \mathcal{H},$$

$$\left\langle \boldsymbol{h} \times \boldsymbol{n}_{\Omega^a} \,, \boldsymbol{a}'\right\rangle_{\Gamma_{\mathrm{m}}} - \left(\nu\, \mathbf{curl}\, \boldsymbol{a} \,, \mathbf{curl}\, \boldsymbol{a}'\right)_{\Omega^a} = 0, \quad \forall \boldsymbol{a}' \in \mathcal{A}.$$

t-*a*-formulation

$$\left\langle w\, \rho\, \mathbf{curl}\, \boldsymbol{t} \,, \mathbf{curl}\, \boldsymbol{t}'\right\rangle_{\Gamma_w} + \left\langle w\, \partial_t \boldsymbol{a} \,, \mathbf{curl}\, \boldsymbol{t}'\right\rangle_{\Gamma_w} = 0, \quad \forall \boldsymbol{t}' \in \mathcal{T},$$

$$\left\langle w\, \mathbf{curl}\, \boldsymbol{t} \,, \boldsymbol{a}'\right\rangle_{\Gamma_w} - \left(\nu\, \mathbf{curl}\, \boldsymbol{a} \,, \mathbf{curl}\, \boldsymbol{a}'\right)_{\Omega_a} = 0, \quad \forall \boldsymbol{a}' \in \mathcal{A}.$$

These are mixed formulations, perturbed saddle point problems:

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle, & \forall v \in V, \\ b(u,q) - c(p,q) = \langle g, q\rangle, & \forall q \in Q, \end{cases} \quad \text{or} \quad \begin{pmatrix} \mathbf{A} & \mathbf{B}^{\mathsf{T}} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}.$$

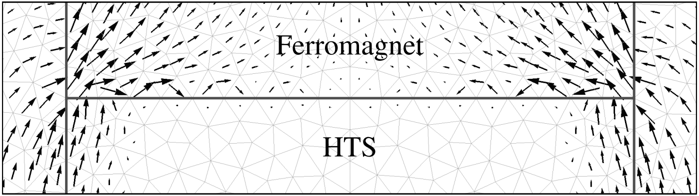
$\Rightarrow$ Compatibility conditions to ensure numerical stability.

D. Boffi, F. Brezzi, et al., Mixed FE methods and applications, Springer, 2013.

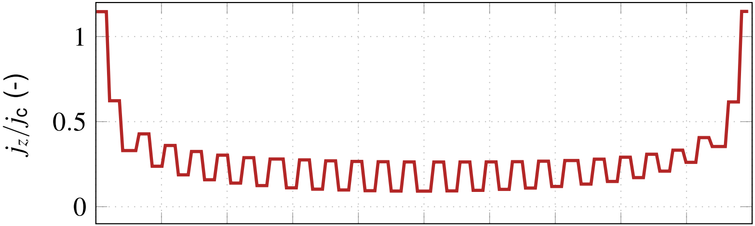
Numerical oscillations

If function spaces do not satisfy the compatibility conditions...

h-a-formulation First-order functions for $\boldsymbol{h}$ and $\boldsymbol{a}$:



t-a-formulation First-order functions for $\boldsymbol{t}$ and $\boldsymbol{a}$:



Compatibility conditions

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^\mathsf{T} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}.$$

The solution is stable, i.e., $\|\boldsymbol{u}\|_V + \|\boldsymbol{p}\|_Q \leq C(\|\boldsymbol{f}\|_{V'} + \|\boldsymbol{g}\|_{Q'})$,
if $\exists \alpha, \beta, \gamma > 0$ (strictly) such that

$$\boldsymbol{v}^\mathsf{T}\mathbf{A}\boldsymbol{v} \geq \alpha\|\boldsymbol{v}\|_V^2,\ \forall \boldsymbol{v} \in \ker(\mathbf{B}) \qquad \text{(coercivity of } \mathbf{A}\text{)},$$

$$\boldsymbol{q}^\mathsf{T}\mathbf{C}\boldsymbol{q} \geq \gamma\|\boldsymbol{q}\|_Q^2,\ \forall \boldsymbol{q} \in \ker(\mathbf{B}^\mathsf{T}) \qquad \text{(coercivity of } \mathbf{C}\text{)},$$

$$\inf_{\boldsymbol{q} \in (\ker(\mathbf{B}^\mathsf{T}))^\perp} \sup_{\boldsymbol{v} \in V} \frac{\boldsymbol{q}^\mathsf{T}\mathbf{B}\boldsymbol{v}}{\|\boldsymbol{q}\|_Q\|\boldsymbol{v}\|_V} \geq \beta > 0 \qquad \text{(inf-sup condition)}.$$

In our case, the inf-sup condition is the most restrictive.

D. Boffi, F. Brezzi, et al., *Mixed FE methods and applications*, Springer, 2013.

Inf-sup test

The inf-sup condition is not easy to check analytically.

$\Rightarrow$ We perform a numerical inf-sup test.

On progressively refined meshes, for given function spaces:

1. Define suitable norms.
2. Extract matrices $\mathbf{B}$, $\mathbf{N}_V$, and $\mathbf{N}_Q$, from the FE assembly, with

$$\|\boldsymbol{v}\|_V^2 = \boldsymbol{v}^{\mathsf{T}}\mathbf{N}_V\boldsymbol{v},$$
$$\|\boldsymbol{q}\|_Q^2 = \boldsymbol{q}^{\mathsf{T}}\mathbf{N}_Q\boldsymbol{q}.$$

3. Solve the eigenvalue problem

$$\left(\mathbf{B}\mathbf{N}_V^{-1}\mathbf{B}^{\mathsf{T}}\right)\boldsymbol{q} = \lambda\mathbf{N}_Q\boldsymbol{q}.$$

Lowest non-zero eigenvalue $=$ square of the inf-sup value β^δ.

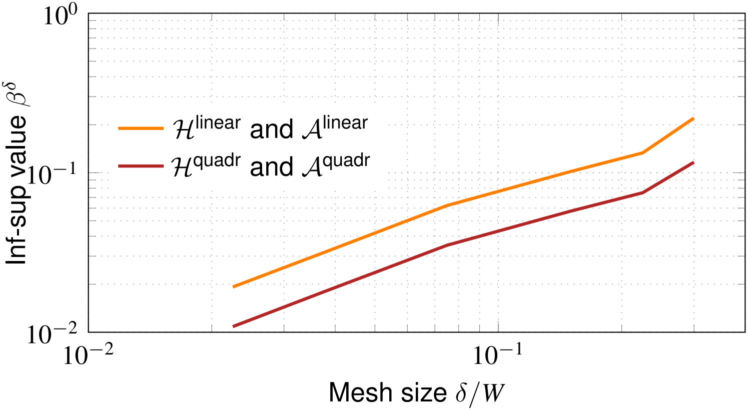
$\Rightarrow$ How does β^δ behave when the mesh is refined?

- It tends to zero $\Rightarrow$ unstable,
- It is bounded from below $\Rightarrow$ stable.

D. Chapelle, K.-J. Bathe, *The inf-sup test*, C&S 47, 1993.

h-a-formulation Unstable choices

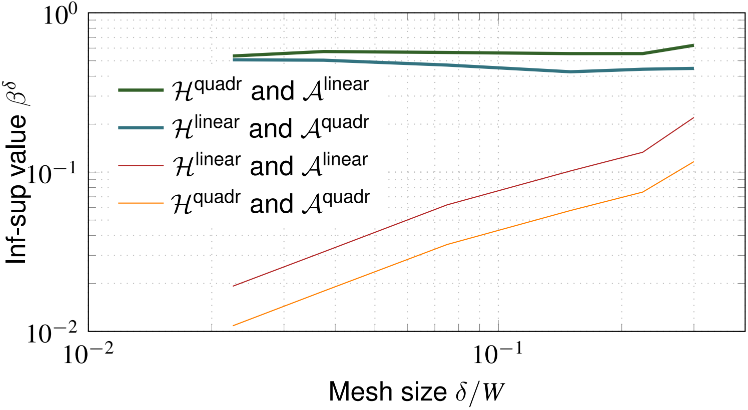
Linear or quadratic elements for both $\boldsymbol{h}$ and $\boldsymbol{a}$ $\Rightarrow$ Unstable.



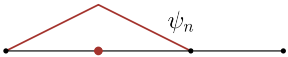
h-a-formulation Stable choices

One way to stabilize the problem:

$\Rightarrow$ Increase the discretization order of one field ($\boldsymbol{h}$ or $\boldsymbol{a}$).

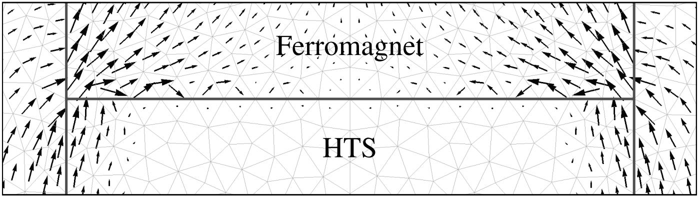


Increasing the order on the coupling interface only is sufficient.

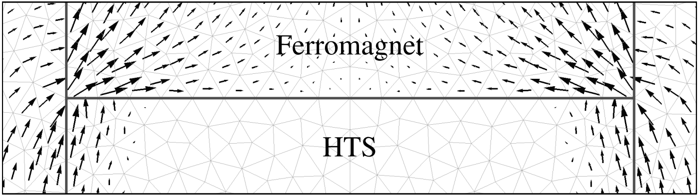


h-a-formulation Stabilization

- First-order functions for $\boldsymbol{h}$ and $\boldsymbol{a}$ (inf-sup KO):

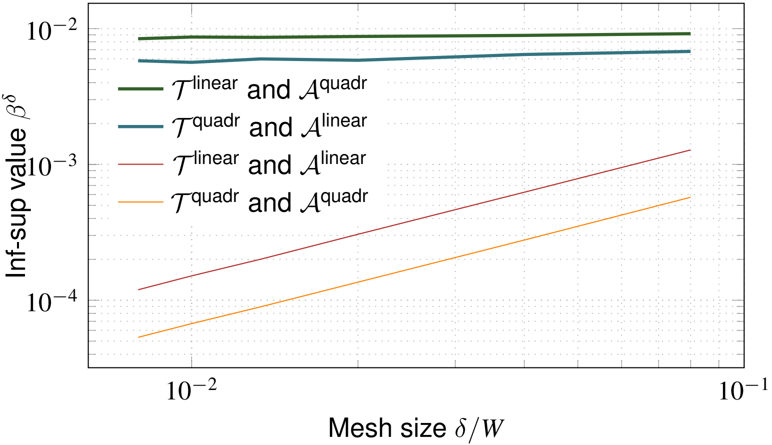


- Second-order for $\boldsymbol{a}$, first-order for $\boldsymbol{h}$ (inf-sup OK):



t-a-formulation - Inf-sup test

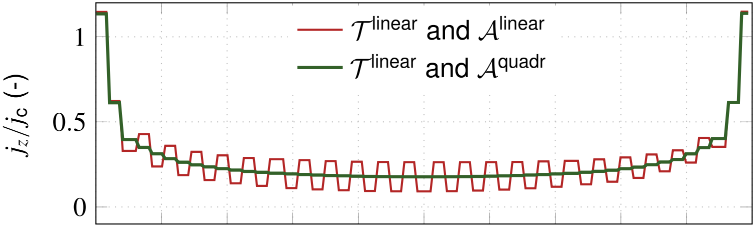
Here, the space for $\boldsymbol{t}$ is analogous to $H^{-1/2}(\Gamma_w)$ and requires special care when defining the norm ("mesh-dependent norm").



Similar conclusions: increase the order of one function space.

t-a-formulation - Stabilization

Example for a 2D case, current density along the tape:



NB: With the choice $\mathcal{T}^{\text{quadr}}$ and $\mathcal{A}^{\text{linear}}$, Newton-Raphson faces convergence difficulties.

Same observation than Berrospe, et al., SUST, vol.32, no.6, p.065003 (2019).

Conclusion and application

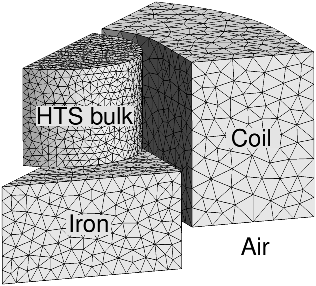
Two coupled formulations

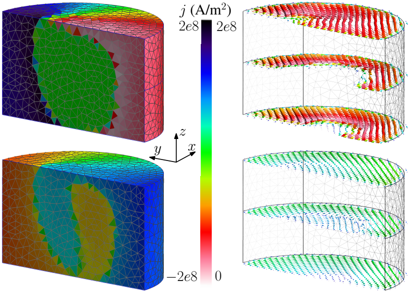
- Modeling HTS and FM efficiently ⇒ h-a-formulation
- Modeling HTS tapes can be done with a t-a-formulation

Both formulations are mixed ⇒ Inf-sup condition.

By enriching one of the two spaces, stability is guaranteed.

Extension to 3D: Bulk magnetization with h-a-formulation ($\boldsymbol{h}$-ϕ)





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